An Edge Partition Problem
Concerning
Perfect Matches
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Spanning Trees in Linear Polygonal Chains

Edward J. Farrell
Michael L. Gargano
Louis V. Quintas
Michael L. Gargano is Professor and quondam Chair of Computer Science at Pace University’s New York City campus. His interests include artificial intelligence (in particular, neural networks and genetic algorithms), discrete mathematics (in particular, graph theory), computer theory, and business applications of quantitative computer models. Mike’s Ph.D. is in combinatorial mathematics.

Professor Gargano has over ten years of real-world business experience and over twenty years of teaching experience (including computer science, mathematics, and management science). Mike enjoys working with students doing research, giving talks, presenting research results, and writing papers. His current research interests are applications of neural networks and genetic algorithms and logic graphs.

Louis V. Quintas studied mathematics at Columbia University and the City University of New York. Since 1967 he has been Professor of Mathematics at Pace University, where he received Pace’s Outstanding Teacher Award in 1975 and the Dyson Society of Fellows Class of 1995 was named in his honor. He has written research articles on functional equations, algebraic topology, combinatorial geometry, graph theory, and on chemical applications of random graphs. He has also published textbooks on linear programming, statistics, and word processing. He has co-chaired and co-edited the proceedings of three international conferences on combinatorial mathematics. He is a Fellow of the New York Academy of Sciences and has chaired its mathematics section. He attributes his love of mathematics to Howard Levi, Fred Supnick, and Alan J. Hoffman, the latter being his doctoral sponsor at the Graduate Center, CUNY.
AN EDGE PARTITION PROBLEM
CONCERNING
PERFECT MATCHINGS

Edward J. FARRELL\textsuperscript{1,*}
Michael L. GARGANO\textsuperscript{2} and Louis V. QUINTAS\textsuperscript{3}
\textsuperscript{1}The University of the West Indies
St. Augustine, TRINIDAD
farrell@centre.uwi.tt

\textsuperscript{2,3}Pace University, New York, NY, 10038 U.S.A
mgargano@pace.edu    lquintas@pace.edu

Abstract

Let $G$ be a graph. An edge $e$ in $G$ is \textit{matchable}, if there exists a perfect matching in $G$ that contains $e$. $G$ is called \textit{totally matchable} if every edge in $G$ is matchable.

We determine sufficient conditions for a graph $G$ to be totally matchable and give a characterization of such graphs. Constructions of graphs having various proportions of matchable edges are also given.

1. Introduction

Let $G = (V(G), E(G))$ be a graph of order $n$ and size $t$.

A \textit{perfect matching} in $G$ is a set of disjoint edges such that the union of the vertices of these edges is all of $V(G)$. An edge $e$ in $G$ is \textit{matchable}, if there exists a perfect matching in $G$ that contains $e$. Since every edge in $G$ is either matchable or not, the edge set of $G$ is partitioned into two subsets, with the empty set $\emptyset$ allowed. The proportion of matchable edges to all edges is either 0 or some value in the interval $[n/2t, 1]$. Clearly, if $G$ has a perfect matching, then $G$ must have even order.

The graph $G$ is called \textit{totally matchable} if every edge in $G$ is matchable.

We determine sufficient conditions for $G$ to be totally matchable, comment on constructing totally matchable graphs, and then give a characterization of such graphs. Graphs with various proportions of matchable edges are discussed.

\*Pace University Visiting Research Scholar 2001-2002

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2. Totally Matchable Graphs

The following are easily verifiable examples of totally matchable graphs.

(i) The complete graph of even order $K_{2m}$, for $m \geq 1$.
(ii) The Petersen graph.
(iii) The complete bipartite graph $K_{m,m}$, for $m \geq 1$.
(iv) The 2-mesh $M(2, b)$, for $b \geq 1$.
(v) The $n$-cube, for $n \geq 1$.
(vi) The cycle of even order $C_{2m}$, for $m \geq 2$.
(vii) The wheel of order $2m$, for $m \geq 2$.

A factor of a graph $G$ is a (vertex) spanning subgraph of $G$. A factor for which every member is $r$-regular is called an $r$-factor. Since a 1-factor is a perfect matching we can use these terms interchangeably. The union of graphs $G$ and $H$ is the graph $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$. If $V(G) = V(H)$ and $E(G) \cap E(H) = \emptyset$, then $G \cup H$ is called the edge sum of $G$ and $H$. If $G$ is expressed as the edge sum of factors of $G$, then this sum is called a factorization of $G$. If there exists a factorization of $G$ such that each factor is a 1-factor, then $G$ is said to be 1-factorable.

The Kuratowski graph, $K_{3,3}$ is 1-factorable. The wheel $W_n$ is not 1-factorable, however if $n$ is even, $W_n$ can be expressed as the union of 1-factors. Clearly, if $G$ is 1-factorable, then $G$ is a union of 1-factors, but not necessarily conversely.

**Theorem 2.1** If $G$ is a union of 1-factors, then $G$ is totally matchable.

**Proof.** Since $G$ is a union of 1-factors, every edge in $G$ is in a 1-factor. ■

**Corollary 2.2** If $G$ is 1-factorable, then $G$ is totally matchable.

**Proof.** Since $G$ is 1-factorable, $G$ is the union of 1-factors. ■

**Corollary 2.3** The graphs $K_{2m}$, $K_{m,m}$, $Q_n$, $C_{2m}$, and $W_{2m}$ are totally matchable.

**Proof.** The graphs $K_{2m}$, $K_{m,m}$, $Q_n$, and $C_{2m}$ are 1-factorable. The graph $W_{2m}$ is a union of 1-factors. ■

**Theorem 2.4** If $G$ has a pendant edge and $G$ is not $K_2$, then $G$ is not totally matchable.

**Proof.** Every perfect matching must contain every pendant edge. No edge adjacent to a pendant edge can be in a perfect matching. Thus, except for $K_2$, a graph with a pendant edge cannot have every edge covered with some perfect matching. ■

**Corollary 2.5** The only nontrivial tree that is totally matchable is $K_2$.

**Proof.** A tree other than $K_2$ has at least one edge incident to a pendant edge. ■

**Corollary 2.6** $G$ is unicyclic and totally matchable if and only if $G = C_{2m}$.

**Proof.** If $G$ is unicyclic and totally matchable, then because of the latter, $G$ has even order. From the theorem, $G$ cannot have any pendant edges. Thus, $G = C_{2m}$. 

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Conversely, an even order cycle is 1-factorable, thus $C_{2m}$ is totally matchable. ■

A graph is **edge-transitive**, if for each pair of edges $e_1, e_2$ there is an automorphism of $G$ that sends $e_1$ to $e_2$.

**Theorem 2.7** Let $G$ be edge transitive, then $G$ is totally matchable if and only if $G$ has a perfect matching.

**Proof.** If $G$ is totally matchable, then by definition $G$ has a perfect matching.

If $G$ has a perfect matching and $G$ is edge transitive, then every edge in $G$ is the image of some edge in the perfect matching and the mapping that does this carries the perfect matching to a perfect matching. ■

**Theorem 2.8** If $G$ has a cut vertex, then $G$ is not totally matchable.

**Proof.** Let $v$ be a cut vertex of $G$ and assume $G$ is totally matchable. Then, $G$ has even order and every edge in $G$ of the form $(v, x)$ is in some perfect matching $M_x$ of $G$. Let $(v, x)$ be such an edge and $H_x$ the component of $G - v$ that contains $x$. Then, $G - v = H_x \cup H$, where $H_x \cap H = \emptyset$. It follows that $M_x - (v, x) - E(H_x)$ is a perfect matching of $H$. Thus, $H$ has even order. Since $v$ is a cut vertex, there exists an edge $(v, y)$ with $y \in V(H)$ and an $M_y$ a perfect matching of $G$ that contains $(v, y)$. Then, $M_y - (v, y) - E(H)$ is a perfect matching of $H_x$ and $H_x$ has even order. Thus, $H_x \cup H$ has even order. However, since $H_x \cup H = G - v$, we have $G$ has odd order. This is a contradiction. ■

**Corollary 2.9** If $G \not= K_2$ has a bridge, then $G$ is not totally matchable.

**Proof.** If $G \not= K_2$ has a bridge, then $G$ has a cut vertex. ■

A Hamilton path $x \rightarrow y$ is a sequence of vertices of $G$, starting with vertex $x$ and ending at vertex $y$, of the form $x_1, x_2, \ldots, x_n$ where $x_1 = x$, $x_i$ is adjacent to $x_{i+1}$, and $x_n = y$, in which each vertex of $G$ appears exactly once. A Hamilton cycle is a Hamilton path such that the first and last vertex coincide, that is, $x_1, x_2, \ldots, x_n, x_1$. Such a path is also called a closed Hamilton path. A graph that contains a Hamilton cycle is said to be **Hamiltonian**.

A graph $G$ is

(i) **Hamiltonian-connected**, if for every pair of distinct vertices $u$ and $v$ in $G$ there is a Hamilton path that connects $u$ and $v$,

(ii) **hypohamiltonian**, if it is not Hamiltonian, but $G - v$ is Hamiltonian for every vertex $v$ in $G$ (the Petersen graph is the smallest order hypohamiltonian graph), and

(iii) **strongly Hamiltonian**, if every edge of $G$ is in some Hamiltonian cycle of $G$.

**Theorem 2.10** If $G$ has even order and is strongly Hamiltonian, then $G$ is totally matchable.

**Proof.** Since $G$ is strongly Hamiltonian, every edge of $G$ is in a Hamilton cycle. A Hamilton cycle of even order consists of two edge disjoint perfect matchings, (that
is, an even cycle is 1-factorable). Thus, every edge of $G$ is in a perfect matching. □

Note that the mesh $M(2, n)$ is totally matchable, but is not strongly Hamiltonian.

**Corollary 2.11** If $G$ has even order and is Hamiltonian connected, then $G$ is totally matchable.

**Proof.** A Hamiltonian connected graph is strongly Hamiltonian □

The join of $G$ and $H$ is denoted $G + H$.

**Corollary 2.12** If $G$ has odd order and is strongly Hamiltonian, then $G + K_1$ is totally matchable.

**Proof.** $G + K_1$ has even order. We show that $G + K_1$ is strongly Hamiltonian. We consider the two cases, edges that are in $G$ and edges that are not in $G$.

Let $\{x, y\}$ be in $G \subset G + K_1$. Then, $\{x, y\}$ is in a Hamilton cycle $C$ in $G$. Let $\{y, z\}$ be in $C$ and let $V(K_1) = \{v\}$. Then $(E(C) - \{y, z\}) \cup \{\{y, v\}, \{v, z\}\}$ is a Hamilton cycle in $G + K_1$ that contains $\{x, y\}$.

Let $\{v, w\}$ be any edge incident to $v$, that is, not in $G$, and $\{w, u\}$ be any edge in $G$. Then, there is a Hamilton cycle $H$ in $G$ that contains $\{w, u\}$. The set of edges $(E(H) - \{w, u\}) \cup \{\{w, v\}, \{v, u\}\}$ is a Hamilton cycle in $G + K_1$, that contains $\{v, w\}$.

Therefore, $G + K_1$ is strongly Hamiltonian and by Theorem 2.10 is totally matchable. □

**Theorem 2.13** If $G$ has even order and is hypohamiltonian, then $G$ is totally matchable.

**Proof.** Let $\{x, y\}$ be any edge in $G$. Then, since $G$ is hypohamiltonian, $G - x$ is Hamiltonian. Let $C$ be a Hamiltonian cycle in $G - x$. Then since $C - y$ is an even order Hamilton path, it has a perfect matching $M$. The set $E(M) \cup \{\{x, y\}\}$ is a perfect matching of $G$ that contains $\{x, y\}$. Thus, $G$ is totally matchable. □

Any sufficient condition for a graph $G$ to be strongly Hamiltonian, Hamiltonian connected, or hypohamiltonian each with the appropriate order, will by the preceding theorems imply that $G$ is totally matchable. With respect to this see, Chapter 4 in [1] for a collection of sufficient conditions for a graph to be strongly Hamiltonian, Hamiltonian connected, or hypohamiltonian. The following theorem is a simple illustration of this.

**Theorem 2.14** If $G$ has even order $n$ and for all distinct nonadjacent vertices $u$ and $v$, $\deg u + \deg v \geq n + 1$, then $G$ is totally matchable.

**Proof.** By a theorem of Ore (see [1]), the hypothesis implies $G$ is Hamiltonian connected and by Corollary 2.11, $G$ is totally matchable. □

Some other examples, involving Hamiltonicity (see [1]), which yield totally matchable graphs use:

1. $G^k$, the $k$-th power of $G$, defined as $V(G^k) = V(G)$ and $u, v$ are adjacent in
$G^k$ whenever their distance is at most $k$ in $G$ and

(2) $L(G)$, the line graph of $G$, defined as $V(L(G)) = E(G)$ and two vertices of $L(G)$ are adjacent if and only if their corresponding edges in $G$ are adjacent. $L(L(G))$ is called the iterated line graph of $G$ and is denoted $L^2(G)$.

For a comprehensive paper characterizing forbidden pairs of subgraphs that imply Hamiltonicity and Hamiltonian-connectedness (see [2]).

**Theorem 2.15** If $G$ is totally matchable and $H$ is any graph, then $G \times H$ is totally matchable.

**Proof.** Let $e$ be an edge in $G \times H$ that lies in a copy of $G$ and $M$ a perfect matching in $G$ that contains $e$. Then, in $G \times H$ use the same matching of $G$ in each copy of $G$. This will be a perfect matching in $G \times H$ that contains $e$.

Now let $e'$ be an edge in $G \times H$ that lies in a copy of $H$. Let $M$ be a matching in $G$. Then, using $e'$ and each of its corresponding copies in each copy of $H$ in $G \times H$ together with copies of $M$ in each copy of $G$ excluding those vertices covered by copies of $e'$ will produce a perfect matching of $G \times H$ that contains $e'$.

Therefore, $G \times H$ is totally matchable. $\blacksquare$

The theorem shows that the prism $G \times K_2$, the torus $C_{2m} \times C_k$, and the $n$-cube $Q_{n-1} \times K_2$ are totally matchable.

**Theorem 2.16** If $G_1$ and $G_2$ are both totally matchable and of the same order, then $G_1 + G_2$ is totally matchable.

**Proof.** We consider two cases. First edges which are either in $G_1$ or $G_2$ and then edges which are not.

**Case 1.** Let $e_1$ be an edge in $G_1$ and $e_2$ an edge in $G_2$. Then, since $G_1$ and $G_2$ are both totally matchable, there exist two matchings $M_1$ and $M_2$ where $M_1$ is a perfect matching that contains $e_1$ and $M_2$ is a perfect matching that contains $e_2$. Then, $M_1 \cup M_2$ is a perfect matching of $G_1 + G_2$ that contains $e_1$ and $e_2$.

**Case 2.** Let $e$ be a connecting edge between $G_1$ and $G_2$ in $G_1 + G_2$. The set of connecting edges form a $K_{m,m}$ where $m$ is the common order of $G_1$ and $G_2$. Since $K_{m,m}$ is totally matchable, let $M$ be a perfect matching of $K_{m,m}$ that contains the edge $e$. Since this matching contains all of the vertices of $G_1 + G_2$, the matching $M$ is a perfect matching of $G_1 + G_2$ that contains $e$.

Thus, $G_1 + G_2$ is totally matchable. $\blacksquare$

In Figure 2.1 we show two small order graphs that are totally matchable but are neither Hamiltonian connected nor hypohamiltonian. Using such examples and trivial modifications, these graphs can be extended to various other totally matchable graphs (for example, the second is a trivial extension of the first) that are neither Hamiltonian connected nor hypohamiltonian.
A hexagonal system is a 2-connected planar graph whose every interior face is bounded by a hexagon. Hexagonal systems are of interest in mathematical chemistry and in such a context are called normal, if every edge of the system is in a perfect matching. That is, normal hexagonal systems are totally matchable. Zhang and Chen [4] proved the following theorem, restated in our language.

Theorem 2.17 A hexagonal system $H$ is totally matchable if and only if for any hexagon $h$ of $H$ there is a perfect matching $M$ such that $h$ is an $M$-alternating cycle.

We close this section with a characterization of totally matchable graphs.

Theorem 2.18 Let $G$ be a graph. Then, $G$ is totally matchable if and only if for each edge $e$ of $G$ there exists a perfect matching $M$ of $G$ and an automorphism $\sigma$ of $G$ such that $\sigma(e)$ is in $M$.

Proof. If $G$ is a totally matchable graph, then by definition, for each edge of $G$ there exists a perfect matching of $G$ that contains $e$. The identity automorphism satisfies the automorphism condition.

Now let $e$ be an edge in $G$ and $M$ a perfect matching of $G$. Let $\sigma$ be an automorphism of $G$ such that $\sigma(e)$ is in $M$. The inverse of $\sigma$ maps $\sigma(e)$ to $e$ and carries $M$ to a perfect matching of $G$ that contains $e$. Thus, $G$ is totally matchable.

Problem 1. Given that a graph $G$ has a perfect matching, what are necessary and sufficient conditions for $G$ to be totally matchable?

3. The proportion of matchable edges

Let $\pi(G)$ denote the proportion of matchable edges to all edges of a graph $G$. Note that $\pi(G) = 0$ if and only if $G$ is does not have a perfect matching and $\pi(G) = 1$ if and only if $G$ is totally matchable.

The following two theorems are easily verified.

Theorem 3.1 Let $G$ be an even cycle of order $n$ with a chord joining two vertices an even distance apart, then

$$\pi(G) = \frac{n}{n+1} \quad \text{and} \quad \lim_{n\to\infty} \pi(G) = 1.$$
The corona of graph $H$ with graph $K$ is defined as the graph $H \circ K$ obtained by taking one copy of $H$ having order $n$ and $n$ copies of $K$ then joining the $i$th-vertex of $H$ with each vertex of the $i$th-copy of $K$ (see [3]).

**Theorem 3.2** Let $G$ be the graph $K_n \circ K_1$, the corona of $K_n$ with $K_1$, then

$$\pi(G) = \frac{n}{\binom{n}{2}} + n$$

and

$$\lim_{n \to \infty} \pi(G) = 0.$$

![Figure 3.1](image1.png)

**Figure 3.1** A graph illustrating Theorem 3.1 with $n = 10$.

![Figure 3.2](image2.png)

**Figure 3.2** The graph $K_4 \circ K_1$ illustrating Theorem 3.2 with $n = 4$.

Theorems 3.1 and 3.2 provide a sequence of graphs from which one can obtain graphs whose numerical ratio of matchable edges to the total number of edges, that is, $\pi(G)$, is as close to 1 or 0 as desired, respectively.

The following theorem solves the problem of finding a graph with a given specified value of $\pi(G)$.

**Theorem 3.3** For any positive rational number $a/b$ (0 < $a$ < $b$) there exists a graph $G$ such that $\pi(G) = a/b$.

**Proof.** If $b - a \leq \left(\frac{a}{2}\right)$ and $a \neq 1$, let $t = b - a$ and $G$ be any graph with $a$ vertices and $t$ edges. Then, $G \circ K_1$, the corona of $G$ with $K_1$, has exactly $a$ matchable edges and $t$ nonmatchable edges. Thus, $\pi(G \circ K_1) = \frac{a}{a + t} = \frac{a}{b}$.

If $b - a > \left(\frac{a}{2}\right)$ or $a = 1$, then we can multiply the numerator and denominator of $\frac{a}{b}$ by $b^2$ so that

$$\frac{a}{b} = \frac{ab^2}{b^3}$$

and

$$b^3 - ab^2 \leq \left(\frac{ab^2}{2}\right).$$

Consequently we can use the preceding construction to obtain a graph with $\frac{ab^2}{b^3} = \frac{a}{b}$ proportion of matchable edges. □
Note that the graphs in Theorem 3.3 are not necessarily connected.

**Problem 2.** Given positive integers $a, b$ with $a \leq b$, for what $a$ and $b$ does there exist a connected graph $G$ of size $b$ such that $G$ has a matchable edges?

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**References**


SPANNING TREES IN LINEAR POLYGONAL CHAINS

Edward J. Farrell

Michael L. Gargano and L.V. Quintas

1Department of Mathematics
The University of The West Indies, St. Augustine, TRINIDAD
ej_farrell@hotmail.com

2Computer Science Department
3Department of Mathematics
Pace University, 1 Pace Plaza, New York NY, 10038, U.S.A
mgargano@pace.edu lquintas@pace.edu

Abstract

Results are given for counting spanning trees in general linear polygonal chains. In the case of regular chains (chains in which the cells are isomorphic), explicit formulae are given for the number of spanning trees and for the number of forests with exactly two components.

1. The Basic Definitions

The graphs considered here are finite and have neither loops nor multiple edges. We refer the reader to Harary [6], for the standard definitions in Graph Theory.

Definitions

Let G and H be graphs with at least one edge. Let uv and xy be edges in G and H respectively. We identify the edge uv with the edge xy by identifying nodes u and x and nodes v and y, and then replace the double edge formed with a single edge. This defines the operation of "sticking" (G, uv) to (H, xy).

The graph formed by sticking (in sequence), n disjoint cycles, each with at least four nodes, using only edges incident to nodes of valency two, is called a linear polygonal chain, with n cells. In this graph, each of (n-2) cycles will be stuck to exactly two cycles, and each of the remaining two cycles, will be stuck to exactly one other cycle. Polygonal chains are of interest to researchers in Mathematical Chemistry (for example, see Gutman [5]). We will denote by $L_{r_1, r_2, r_3, \ldots, r_n}$, the linear polygonal chain consisting of a sequence of $r_i$-cycles, where $r_i \geq 4$ and $i = 1, 2, \ldots, n$.

Definitions

(i) A chain is a tree with nodes of valencies 1 and 2 only.
The chain with n nodes will be denoted by $P_n$. (Notice that a chain is called a path, when it occurs as a proper subgraph of a graph)

(ii) Let G be a graph with at least two nodes x and y. We add the chain $P_n$ (n ≥ 1) to G, at nodes x and y (called nodes of attachment) by identifying one endnode of $P_n$, with node x and the other endnode, with node y (The chain $P_n$ then becomes a path in the resulting graph).

(iii) The short ladder $S_n$ is the graph formed by joining the corresponding nodes of two equal chains $P_n$. Assuming the standard drawing of $S_n$.

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we refer to the two chains as "upper" and "lower" chains; and to their edges, as "upper" and "lower" edges. We will refer to the connecting edges, as link edges; their incident nodes (original nodes of \( P_n \)), link nodes. 

(iv) The regular linear polygonal chain \( L_{r,n} \) with \( n \) cells, is the homeomorph of \( S_n \), formed by inserting \((r-4)/2\) new nodes in each upper and lower edge, when \( r \) is even; and, when \( r \) is odd; by inserting \((r-3)/2\) nodes in each upper edge and \((r-5)/2\) nodes (i.e. one less node) in each lower edge. In \( L_{r,n} \), each cell is an \( r \)-cycle. Therefore we denote \( L_{r,n} \), simply by \( L_n \), when \( r \) is fixed.

(v) The two cells of \( L_{r,n} \), which contain only one common link edge (i.e. the ones at its "ends"), are its terminal cells. The link edge of a terminal cell, which is incident to nodes of valency two, is called a terminal edge.

It can be easily shown that \( L_{r,n} \) has \( n(r-2)+2 \) nodes and \( n(r-1)+1 \) edges. Conventionally, we take \( L_{r,0} \) to be an edge.

We will derive results which can be used for counting spanning trees in linear polygonal chains. In the case of the regular linear polygonal chains, we will derive explicit formulae for the number of spanning trees and also for the number of spanning forests with exactly two components.

2. Spanning Trees in Linear Polygonal Chains

Let \( G \) be a graph, and \( e \) an edge in \( G \). We can put the spanning trees in \( G \) into two disjoint classes: (i) those containing \( e \) and (ii) those which do not contain \( e \). The spanning trees which do not contain \( e \), are spanning trees of the graph \( G' \), obtained from \( G \) by deleting \( e \). The spanning trees which contain \( e \), are spanning trees of the incorporated graph \( G^* \) obtained from \( G \), by requiring \( e \) to belong to every spanning tree of \( G^* \). A convenient way of doing this; is to shrink the edge \( e \). This is equivalent to identifying the nodes at the ends of \( e \). Since every spanning tree of \( G^* \) contains all the nodes of \( G^* \), then, by viewing the new node formed by the identification process, as the entire subgraph consisting of \( e \) and its end nodes, we can see that all the spanning trees of \( G^* \) must contain \( e \).

Since we are interested only in counting the spanning trees in \( G \), it will be unnecessary to keep track of the edges "contained" in the new node. Thus, we can identify several of these nodes themselves and lose no information about the number of spanning trees in \( G \). Hence all the nodes in \( G^* \) can be treated as ordinary nodes.

Let \( \Gamma(G) \) denote the number of spanning trees in the graph \( G \). Then, our discussion leads to the following result (which is also given in Farrell[2]).

**Lemma 1**

Let \( G \) be a graph, and \( e \) an edge in \( G \). Let \( G' \) be the graph obtained from \( G \) by deleting \( e \) and \( G^* \), the graph obtained from \( G \) by identifying the nodes at the ends of \( e \). Then

\[
\Gamma(G) = \Gamma(G') + \Gamma(G^*).
\]
This lemma yields an algorithm for counting the spanning trees in a graph. By recursive applications of the theorem to \( G' \) and \( G^* \); and to subsequent graphs, we can obtain the number of spanning trees in \( G \) in terms of the number of spanning trees in smaller graphs \( H_i \) for which \( \Gamma(H_i) \) are known. For obvious reasons, we can ignore intermediate graphs which are disconnected. Also, any loops formed by the identification process can be immediately deleted; since a loop will represent the final edge required to complete a cycle in the incorporated graph. Thus, the identification process is the same as the process described by Read[7] for chromatic polynomials. This algorithm is originally due to Feussner ([3], [4]).

We will now obtain a generalization of Lemma 1, from which a more general algorithm will be deduced. Let us denote by \( G(n) \), the graph obtained from a graph \( G \) by adding a chain of length \( n \) (see (iii)), at nodes \( x \) and \( y \). Clearly, any spanning tree in \( G(n) \) must either contain the entire chain or all but one of the \( n \) edges. From a generalization our discussions above, it can be easily seen, that the spanning trees which contain the entire chain, are spanning trees of the graph \( G^* \) obtained from \( G(n) \) by identifying the nodes of attachment, and then removing the "loop" containing all the edges of \( P_{n+1} \). Therefore, for purposes of counting the spanning trees, the resulting graph will be isomorphic to \( G^* \).

For each edge \( e \) of \( P_{n+1} \), the spanning trees which do not contain \( e \), will be the spanning trees of the graph obtained from \( G(n) \), by deleting \( e \). However, as discussed above, the remaining edges of \( P_{n+1} \) can then be shrunk into the nodes of attachment, without affecting the count. Therefore, for purposes of counting spanning trees, the resulting graph will be isomorphic to \( G \).

Our discussion leads to the following generalization of Feussner's result (see [2]).

**Theorem 1**

Let \( G(n) \), the graph obtained from a graph \( G \) by adding a chain of length \( n \). Then

\[
\Gamma(G(n)) = n\Gamma(G) + \Gamma(G^*),
\]

where \( G^* \) is the graph obtained from \( G \), by identifying the nodes of attachment. (As expected, when \( n = 1 \), this theorem reduces to the result of the lemma.)

We can easily obtain an extension of Theorem 1. Let \( G(m, n) \) denote the graph obtained from a graph \( G \), by adding at nodes \( x \) and \( y \), two chains; one of length \( m \) and the other of length \( n \). It is not difficult to deduce the following result, by two applications of Theorem 1.

**Theorem 2**

\[
\Gamma(G(m, n)) = mn\Gamma(G) + (m + n)\Gamma(G^*).
\]

Theorems 1 and 2, provide useful tools for counting spanning trees in linear polygonal chains. In these linear chains, the number of spanning trees will be independent of the edges used in the "sticking" process. Thus, the results will depend only on the number of edges in the cells, and the number of cells.
We will now consider the general linear polygonal chain
$L_{r_1, r_2, r_3, \ldots, r_n}$. Apply Theorem 1 to $L_{r_1, r_2, r_3, \ldots, r_n}$, by taking $G$ to be
$L_{r_1, r_2, r_3, \ldots, r_{n-1}}$ and the attached chain, to be a chain of length $(r_{n-1})$ attached to
the link nodes of a terminal cell. Then, the graph $G^*$ will be
$L_{r_1, r_2, r_3, \ldots, (r_{n-1}-1)}$. The theorem then yields the following result.

**Lemma 2**

$$
\Gamma(L_{r_1, r_2, r_3, \ldots, r_n}) = (r_{n-1})\Gamma(L_{r_1, r_2, r_3, \ldots, r_{n-1}}) + \\
\Gamma(L_{r_1, r_2, r_3, \ldots, r_{n-1}, (r_n-1)}).
$$

This recurrence, when used recursively with Theorem 1, can be used to find the number of spanning trees in any linear polygonal chain. We illustrate this, by deducing explicit results for all linear polygonal chains, with up to four cells.

Notice that if we put $n = 1$, in Lemma 2, then the graph $G$ will have zero cells. It would therefore be an edge. The attached chain will be of length $r_1-1$. In this case, $G^*$ will be the graph obtained from $G$, by identifying the nodes at the ends of the edge. Therefore $G^*$ will be a node. Trivially,

$$
\Gamma(G) = \Gamma(G^*) = 1.
$$

Therefore, the lemma yields

$$
\Gamma(L_{r_1}) = \Gamma(K_1) + (r_1-1)\Gamma(K_2) = 1 + (r_1-1) = r_1,
$$

as is otherwise well-known.

Let us use Lemma 2, with $n = 2$. Then $L_{r_1, r_2, r_3, \ldots, r_{n-1}} (= G)$ will be an
$r_1$-cycle and $L_{r_1, r_2, r_3, \ldots, r_{n-1}, (r_n-1)} (= G^*)$ will be an $(r_1-1)$-cycle. Therefore, we get

$$
\Gamma(L_{r_1, r_2}) = \Gamma(L_{r_1, r_2-1}) + (r_2-1)\Gamma(L_{r_1, r_1}).
$$

$$
= r_1-1 + (r_2-1)r_1.
$$

Hence we obtain the following result.

**Lemma 3**

$$
\Gamma(L_{r_1, r_2}) = r_1r_2 - 1.
$$

Let us use Lemma 2, with $n = 3$. Then $L_{r_1, r_2, r_3, \ldots, r_{n-1}} (= G)$ will be
$L_{r_1, r_2}$; and $L_{r_1, r_2, r_3, \ldots, r_{n-1}, (r_n-1)} (= G^*)$ will be $L_{r_1, r_2-1}$. Therefore, we get

$$
\Gamma(L_{r_1, r_2, r_3}) = \Gamma(L_{r_1, r_2-1}) + (r_3-1)\Gamma(L_{r_1, r_2}).
$$

$$
= r_1(r_2-1) - 1 + (r_3-1)(r_1r_2 - 1),
$$

using Lemma 3.

On simplification, we obtain the following result.

**Lemma 4**

$$
\Gamma(L_{r_1, r_2, r_3}) = r_1r_2r_3 - r_1 - r_3.
$$

It is not difficult to extended the analysis to linear polygonal chains with four cells. The result is given in the following lemma.
Lemma 5
\[ \Gamma(L_{r_1, r_2, r_3, r_4}) = r_1r_2r_3r_4 - r_1r_2 - r_1r_4 - r_3r_4. \]

Clearly, this technique can be used to obtain explicit results for linear polygonal chains with any finite number of cells.

3. Spanning Trees in Regular Linear Polygonal Chains

For the regular linear polygonal chain, \( r_i = r \), for all \( i \). In this case, we write \( L_n \) for \( L_{r, r, r, \ldots, r} \) (\( n \) times). The graph corresponding to \( L_{r, r, r, \ldots, r, r} \) will be denoted by \( L_n^* \). Therefore \( L_n^* \) is the (distorted) polygonal chain, obtained from \( L_n \), by reducing a terminal cell to a (r-1)-cycle.

From Lemma 2, we get
\[ \Gamma(L_n) = \Gamma(L_{n-1}^*) + (r-1)\Gamma(L_{n-1}). \]  \( \ldots (1) \)

By using Lemma 2, for the graph \( L_n^* \), we get
\[ \Gamma(L_n^*) = \Gamma(L_{n-1}^*) + (r-2)\Gamma(L_{n-1}). \]  \( \ldots (2) \)

From Equation (1), we have
\[ \Gamma(L_{n-1}^*) = \Gamma(L_n) - (r-1)\Gamma(L_{n-1}). \]  \( \ldots (3) \)

By substituting (3) into Equation (2), and simplifying, we obtain the following result.

Lemma 6

The number of spanning trees in the \( r \)-regular linear polygonal chain with \( n \) cells satisfies the recurrence
\[ \Gamma(L_n) = r\Gamma(L_{n-1}) - \Gamma(L_{n-2}), \]
with \( \Gamma(L_0) = 1 \) and \( \Gamma(L_1) = r \).

Let us write \( L(t) \) for the generating function for \( \Gamma(L_n) \), with indicator function \( t \). Then the above recurrence yields
\[ L(t) - L_0 - L_1t = r[L(t) - L_0] - tL(t). \]
\[ \Rightarrow L(t)(1 - rt + t^2) = L_0 + L_1t - rL_0 \]  \( \ldots (4) \)

By substituting for \( L_0 \) and \( L_1 \), we get
\[ L(t) = (1 - rt + t^2)^{-1} = \frac{A}{t - \alpha} + \frac{B}{t - \beta}, \]

where \( \alpha \) and \( \beta \) are the roots of \( 1 - rt + t^2 = 0 \). Also \( A = (\beta - \alpha)^{-1} \) and \( B = (\alpha - \beta)^{-1} \).
\[ \Rightarrow L(t) = \frac{1}{\alpha - \beta} \left[ \sum_{n=0}^{\infty} t^n \left( \alpha^{-(n+1)} - \beta^{-(n+1)} \right) \right]. \]  \( \ldots (5) \)

Now, the roots of the quadratic \( 1 - rt + t^2 = 0 \) are \( \frac{r \pm \sqrt{r^2 - 4}}{2} \). Put
\[ \varepsilon = \sqrt{r^2 - 4} \] Then (without loss of generality) \( \alpha = (r + \varepsilon)/2 \) and \( \beta = (r - \varepsilon)/2 \). This implies that \( \alpha - \beta = \varepsilon \). Therefore, Equation (5) yields the following result.
Theorem 2

\[ \Gamma(L_n) = \frac{1}{\varepsilon} \left[ \left( \frac{r + \varepsilon}{2} \right)^{n+1} - \left( \frac{r - \varepsilon}{2} \right)^{n+1} \right]. \]

The short ladder \(S_n\) is the special regular linear polygonal chain \(L_{4,n-1}\), with \((n-1)\)-cells, in which each cell is a 4-cycle. When \(r = 4\), we get \(\varepsilon = 2\sqrt{3}\). Therefore, Theorem 2 yields the following result.

Corollary 2.1

The number of spanning trees in the short ladder \(S_n\) is

\[ \frac{1}{2\sqrt{3}} \left[ (2 + \sqrt{3})^n - (2 - \sqrt{3})^n \right]. \]

This agrees with the result given in Sedlacek[6]; and also in Farrell[1]. The regular linear benzene chain with \(n\) cells, is the special regular linear polygonal chain \(L_{6,n}\). When \(r = 6\), we get \(\varepsilon = 4\sqrt{2}\). Therefore Theorem 2 yields the following result.

Corollary 2.2

The number of spanning trees in the linear benzene chain with \(n\) cells is

\[ \frac{1}{4\sqrt{2}} \left[ (3 + 2\sqrt{2})^{n+1} - (3 - 2\sqrt{2})^{n+1} \right]. \]

4. A Deduction for Certain Spanning Forests With Two Trees

Recall that the graph \(L_n^*\) is the graph in which the entire attached chain \(P_r\) is incorporated. This means that every spanning tree in \(L_n^*\), contains the chain \(P_r\). Clearly then, the nodes at the ends of \(P_r\) must belong to different components of a spanning forest of \(L_n\). Furthermore, any such forest must have exactly two components. This is the only way that the addition of the chain \(P_r\) could yield a spanning tree in \(L_n\). Hence \(\Gamma(L_n^*)\) is also the number of spanning forests in \(L_n\), with two components -each of which contains exactly one of the link nodes.

From Equation (2), we have

\[ \Gamma(L_{n-1}) = \frac{1}{r - 2} \left[ \Gamma(L_n^*) - \Gamma(L_{n-1}^*) \right]. \]

Therefore

\[ \Gamma(L_n) = \frac{1}{r - 2} \left[ \Gamma(L_{n+1}^*) - \Gamma(L_n^*) \right]. \]

By substituting in Equation (1), and simplifying, we obtain the following explicit recurrence for \(\Gamma(L_n^*)\).

\[ \Gamma(L_n^*) = r\Gamma(L_{n-1}^*) - \Gamma(L_{n-2}^*), \]

with \(\Gamma(L_0^*) = 1\) and \(\Gamma(L_1^*) = r - 1\).

Hence we have the following result.
**Lemma 7**

Let $\gamma_n$ be the number of spanning forests in $L_n$, with two components; such that the link nodes of one of its terminal cells belong to different components. Then

$$\gamma_n = r\gamma_{n-1} - \gamma_{n-2},$$

with $\gamma_0 = 1$ and $\gamma_1 = r - 1$.

We can solve the recurrence given in Equation (6), in a similar manner. In this case, Equation (4) becomes

$$L^*(t)[1 - rt + t^2] = L_0^* + L_1^*t - rtL_0^* = 1 - t.$$

$$\Rightarrow$$

$$L^*(t) = (1 - t)(1 - rt + t^2)^{-1}.$$

In this case we get $A = \frac{\alpha - 1}{\beta - \alpha}$ and $B = \frac{\beta - 1}{\alpha - \beta}$ and Equation (5) becomes

$$L^*(t) = \sum_{n} t^n \left[ \frac{\alpha - 1}{\beta - \alpha} \frac{1}{\alpha - (n+1)} + \frac{\beta - 1}{\alpha - \beta} \frac{1}{\beta - (n+1)} \right].$$

Hence, by extracting the coefficient of $t^n$ (and recalling that $\epsilon = \sqrt{r^2 - 4}$), we obtain the following result.

**Theorem 3**

$$\gamma_n = \left( \frac{2 - r - \epsilon}{2} \right)^{n+1} - \left( \frac{2 - r + \epsilon}{2} \right)^{n+1}.$$

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