Constructions of Connected Graphs with a Given Matchable Ratio

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CONSTRUCTIONS OF CONNECTED GRAPHS WITH A GIVEN MATCHABLE RATIO
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Abstract

Given a positive rational number \( \frac{a}{b} (0 \leq a \leq b) \), we identify families of
connected graphs \( G \), such that the ratio of the number of matchable edges to the
total number of edges in \( G \) - denoted by \( \pi(G) \), is \( \frac{a}{b} \). We call \( \pi(G) \), the
matchable ratio of \( G \). For certain kinds of rational numbers, we identify the
smallest connected graphs with the property. This problem was initially
discussed in [1].

1. Introduction

The graphs considered here are finite, and contain neither loops
nor multiple edges. Let \( G \) be a graph. An edge of \( G \) is called matchable, if and
only if it belongs to a perfect matching in \( G \). This paper continues the
investigation into matchable edges started in [1], where we focused on totally
matchable graphs, that is, graphs in which every edge is matchable. We now
turn our attention to graphs which are not totally matchable. These are graphs
which contain non-matchable edges.

Definition

Let \( G \) be a graph with \( b \) edges; i.e. of size \( b \), and with \( a \) matchable
edges. Then the ratio \( \pi(G) = \frac{a}{b} \) is called the matchable ratio of \( G \).

The following are additional definitions, which will apply to the
material that follows.

Definitions

(i) A 1-cycle and a 2-cycle is a vertex and an edge respectively. A
cycle with more than two vertices is called a proper cycle.

In the material that follows, "cycle" will mean "proper cycle", unless
otherwise specified.

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(ii) A graph $G$ is non-matchable (matchable) saturated if and only if no more non-matchable (matchable) edges can be added to it.

(iii) An edge joining two non-adjacent vertices of a cycle is a chord (The 3-cycle has no chords).

(iv) An even $n$-cycle is canonically labeled, if an only if its vertices are labeled in some agreed order, with the consecutive integers from 1 to $n$. For purposes of this paper, we will take the order to be a clockwise.

(v) In a canonically labeled cycle, a chord joining two odd (even) labeled vertices is called odd (even); otherwise it is mixed.

(vi) A graph, consisting of a canonically labeled $r$-cycle $C_r$, with $s$ odd chords and no even chords added, will be denoted by $G_{r,s}$. The subgraph $C_r$ is called its boundary.

(vii) A chain is a tree with nodes of valency 1 and 2 only. A boundary chain of $G_{r,s}$ is any connected subgraph of its boundary.

From the definition, the graph $G_{r,s}$ contains $r \geq 4$ vertices and $r+s$ edges. The graph $G_{r,0}$ is the cycle $C_r$.

In [1] we identified graphs with certain matchable ratios. We also established the existence of a graph for any given matchable ratio (Theorem 3.3). The proof of this result also provides an algorithm for the construction of such graphs. The graphs obtained from this theorem, may be disconnected. It is therefore interesting to be able to construct connected graphs with a given mixing ratio $\frac{a}{b}$. Even more interesting, is the construction of a connected graph having precisely size $b$ and with $a$ matchable edges. In this paper, we give constructions for such graphs. Moreover, we identify the smallest order graphs with this property.

2. Graphs with matchable ratio $a/b$, where $a = 2n$ $(n > 1)$,

$$0 \leq a \leq b \quad \text{and} \quad b - a \leq \binom{n}{2}$$

Lemma 1

In the graph $G_{r,s}$, every chord is non-matchable.

Proof

Figure 1 shows a canonical drawing of $G_{r,s}$. 

52
All chains referred to, will be boundary chains. It can be easily seen that a chain has odd order, if and only if the labels of its end-vertices have the same parity. Let us assume that there exists a perfect matching $M$ containing the (odd) chord $xy$. Now, remove vertices $x$ and $y$ from $G_{r,s}$. Since $x$ and $y$ are odd and there are no even or mixed chords in $G_{r,s}$, we have in $G_{r,s}$ $x$-$y$ vertices $x-1$, $x+1$, $y-1$ and $y+1$ each having valency 1 (see Figure 2) and all other even vertices have valency 2. In particular, the edges $(x-1, x-2), (x+1, x+2), (y-1, y-2)$ and $(y+1, y+2)$ must all be in $M$. 

Figure 1

Figure 2
The edge joining $x+1$ to $x+2$ must belong to $M$. This forces the edge joining $x+3$ to $x+4$ to be in $M$. By continuing in this manner, we get that the edge joining $y-3$ and $y-2$ must belong to $M$. But the edge $(y-1, y-2)$ is in $M$. This is a contradiction; since (by assumption) $M$ is a perfect matching. Thus our assumption is false. The chord $xy$ is non-matchable. Hence the result follows.

**Theorem 1**

Let $r = 2n$ ($n \geq 1$). If $s = \binom{n}{2}$, then the graph $G_{r,s}$ is non-matchable saturated.

**Proof**

From the lemma, all the chords of $G_{r,s}$ are non-matchable. There are $n$ odd labelled vertices in $G_{r,s}$. Any pair of these can be joined to form an odd chord. The number of such pairs is $\binom{n}{2}$. Therefore, when $s$ takes this value, every pair of odd labelled vertices are joined by an edge, so that all odd chords are included.

We must now show that no more non-matchable chords can be added to $G_{r,s}$, that is, every new chord is matchable. Let us add a new chord $xy$. Then $xy$ must either be (i) even or (ii) mixed. Call the resulting graph $G$.

**Case (i) (xy is even)**

Let us remove vertices $x$ and $y$ from $G$. In the resulting graph $G'$, vertices $x-1$ and $x+1$, being odd vertices, will be joined by an edge (being odd labelled vertices); and so too will be vertices $y-1$ and $y+1$. Thus, the resulting graph will contain a new boundary cycle $C_{r-2}$. Since $r$ is even, $r-2$ is even. Thus, $G'$ has a perfect matching; and the chord $xy$ is matchable in $G_{r,s}$.

**Case (ii) (xy is mixed)**

Without loss in generality, we will assume that $x$ is odd and that $y$ is even. Again, let us remove vertices $x$ and $y$ from $G$ to obtain a graph $G'$. Then, $G'$ will contain two boundary chains—one chain connecting vertex $x+1$ (even) to vertex $y-1$ (odd); the other, connecting vertex $x-1$ to vertex $y+1$. Since the chains have endnodes with different parities, they will be even chains; and consequently, have perfect matchings. Hence $G'$ has a perfect matching. Adding the chord $xy$ to this matching, yields a perfect matching in $G_{r,s}$.

We conclude therefore, that no more non-matchable edges can be added to $G_{r,s}$. Hence $G_{r,s}$ is saturated.

**Corollary 1.1**

Let $a = 2n$ ($n \geq 2$) and $b$ be positive integers, with $a \leq b$ and $0 \leq b-a \leq \binom{n}{2}$.

Then the graph $G_{a,b-a}$ is a connected graph such that $\pi(G_{a,b-a}) = \frac{a}{b}$.
Proof
Let the number of added (odd) chords be \( \varepsilon \). Then (from Theorem 1) we get

\[
0 \leq \varepsilon \leq \binom{n}{2}.
\]

Since \( C_a \) has \( a \) edges, total number of edges in \( G \) is \( a + \varepsilon \).

\[
\Rightarrow \pi(G) = \frac{a}{a + \varepsilon}.
\]

But \( \varepsilon = b - a \). Therefore, the result follows.

3. Graphs with matchable ratio \( a/b \), with \( a = 2n + 1 \) ( \( n > 1 \)),

\[0 < a < b \text{ and } b - a \leq \frac{4}{400} \]

**Definition**

The graph \( H_r \) is a canonically labeled \( r \)-cycle, with a chain of length 2, attached to one of its vertices; and with the vertex of valency 2 and 1, labeled \( r + 2 \) and \( r + 1 \) respectively. The graph \( H_{r,s} \) is the labeled graph \( H_r \), with \( s \) odd chords added (See Figure 3).

![Graph H_{r,s}](image)

**Figure 3**

This graph is essentially the graph \( G_{r,s} \), with a chain of length 2 attached to one of its boundary vertices. By definition, the graph \( H_{r,s} \) contains \( r + 2 \) vertices and \( r + 2 + s \) edges.
The following lemma is analogous to Lemma 1.

**Lemma 2**
In the graph $H_{r,s}$, every chord is non-matchable.

**Proof**
In Figure 3, we show a canonical drawing of a labeled graph $H_{r,s}$. The attached chain does not affect the arguments given in the proof of Lemma 1, since the edge joining $r+1$ to $r+2$ must be used in every perfect matching. Hence the result follows.

This lemma implies that we can still add non-matchable odd (or even) chords to the subgraph $G_{r,s}$ of $H_{r,s}$. Also, it is clear that the edge joining the vertex $r+1$ to the boundary is non-matchable. Therefore, $H_{r,s}$ will have $r+1$ matchable edges and $s+1$ non-matchable edges. In $H_{r,s}$ new kinds of edges can join the "external" vertices $r+1$ and $r+2$ to vertices on the boundary of $G_{r,s}$. It is difficult to tell which ones are non-matchable, since the presence of these edges could even spoil the non-matchability of chords. We can however saturate the $G_{r,s}$ subgraph of $H_{r,s}$, so that the resulting graph is saturated with non-matchable chords. This yields the following analogy to Theorem 1.

**Theorem 2**
Let $n$ be an integer greater than 2, and let $a = 2n+1$. Then the graph $H_{a-1,s}$ is saturated with non-matchable chords, when $s = \binom{n}{2}$.

The following result is immediate from Theorem 2; and is analogous to Corollary 1.1.

**Corollary 2.1**
Let $a = 2n + 1$ ($n \geq 2$) and $b$ be positive integers, with $a \leq b$ and $0 \leq b - a \leq \binom{n}{2}$. Then the graph $H_{a-1,b-a+1}$ is a connected graph such that

$$
\pi(H_{a-1,b-a+1}) = \frac{a}{b}.
$$

4. **Graphs with matchable ratio $a/b$ in which $b-a$ is not bounded above by $n$**.

We now consider the case in which $0 \leq a < b$ and $b-a$ is not bounded above by $\binom{n}{2}$. In the results above, $b-a$ is bounded above by $\binom{n}{2}$ - the number of chords that can be added to the boundary cycle. This excludes many classes of rational numbers. For example, the rational number $\frac{1}{100}$ is not
covered by the results above, since they all are based on the condition that \( C_a \) is a proper cycle. In fact, the smallest even value of \( a \) is 4 (Corollary 1.1) and the smallest odd value is 5 (Corollary 2.1). Therefore the numerator of the fraction must be at least 4. In this case the fraction will be \( \frac{4}{400} \).

When \( b-a \) is bounded, as defined above, we have identified a connected graph \( G \) of size \( b \) with a matchable edges, such that \( \pi(G) = \frac{a}{b} \). However, for some rational numbers, it will be impossible to find a graph; connected or not, with this property. For example, for the rational number \( \frac{1}{100} \), one would have to find a graph with 100 edges in which exactly one edge belongs to a perfect matching. No such graph exists. We will therefore consider the related problem of finding a connected graph \( G \), such that \( \pi(G) = \frac{a}{b} \). This means that there are no restrictions on the size of the graph. Our technique is based on the simple fact that the rational numbers \( \frac{a}{b} \) and \( \frac{ka}{kb} \) are equal, for all non-zero values of \( k \). This will allow us to use the construction given in Sections 2 and 3, since we can always arrange for \( ka \) to be even. We will do better than this. We will identify a smallest order graph obtained by our construction, that is, a smallest order \( G_{r,s} \).

The following result is crucial.

**Lemma 3**

For all positive integers \( a \) and \( b \), with \( b \geq a \), there exists a positive integer \( k \) such that

\[
k b - ka \leq \left( \frac{ka}{2} \right), \quad \text{when } ka \text{ is even. Furthermore, the smallest value of } k \text{ for which the condition holds is } \left\lfloor \frac{2}{a} \left( \frac{4}{a} (b - a) + 1 \right) \right\rfloor.
\]

**Proof**

Since \( b \geq a \), then \( b-a \geq 0 \) and \( \left\lfloor \frac{2}{a} \left( \frac{4}{a} (b - a) + 1 \right) \right\rfloor \) is a positive integer.

Choose \( k \geq \left\lceil \frac{2}{a} \left( \frac{4}{a} (b - a) + 1 \right) \right\rceil \). Then \( \frac{ka}{2} \geq \left( \frac{4}{a} \right) (b - a) + 1. \) From this, we obtain \( \frac{ka}{2} - 1 \geq \left( \frac{4}{a} \right) (b - a) \), which in turn, implies that \( b - a \leq \frac{a}{4} \left( \frac{ka}{2} - 1 \right) \).
Thus \( k(b-a) \leq \frac{ka}{4}\left(\frac{ka}{2} - 1\right) = \left(\frac{ka}{2}\right)^2 \), when \( ka \) is even. Hence there exists such a positive integer \( k \). It can be easily shown that if \( k < \left\lceil \frac{2}{a} \left(\frac{4}{a}(b-a)+1\right) \right\rceil \), the inequality no longer holds. Hence the result follows. □

This lemma identifies the range of values of \( k \) which would make the \( ka \)-cycle (when \( ka \) is even) large enough so that there would be enough "room" to add the necessary number \( (kb-ka) \) of chords. We can now construct a graph \( G \), for which \( \pi(G) = \frac{ka}{kb} \). This is the gist of the following theorem.

**Theorem 3**

Let \( \frac{a}{b} \) be a positive rational number. Then, for \( n = ka \) and

\[
s = k(b-a), \text{ where } k \geq \left\lceil \frac{2}{a} \left(\frac{4}{a}(b-a)+1\right) \right\rceil ,
\]

the graph \( G_{n,s} \) has the property that \( \pi(G_{n,s}) = \frac{a}{b} \), when \( ka \) is even. When \( ka \) is odd, the graph \( H_{n-1,s} \) has the property that \( \pi(H_{n-1,s}) = \frac{a}{b} \).

**Proof**

The result follows immediately from Corollary 1.1, Corollary 2.1 and Lemma 3. □

This theorem gives the range of values of \( k \) for which graphs of the types \( G_{r,s} \) and \( H_{r,s} \) can be constructed with a given matchable ratio.

**Example 1**

Let \( \frac{a}{b} = 1 \). Then \( a = b \). In this case, \( b-a = 0 \), so that \( k = \frac{2}{a} = 1, n = ka = 2 \) and \( s = k(b-a) = 0 \). The resulting graph is \( G_{n,s} = G_{2,0} \); which is a 2-cycle with no chords added. Therefore, the graph is an edge.
Example 2

Let \( \frac{a}{b} = \frac{1}{100} \). Then \( a = 1 \) and \( b = 100 \). In this case, the smallest value of \( k \) is

\[
\left[ \frac{\frac{2}{1} \left( \frac{4}{1} (100 - 1) + 1 \right)}{\frac{1}{1}} \right] = 2(397) = 794. \text{ Therefore } \frac{ka}{kb} = \frac{794}{79400}.
\]

Therefore resulting graph is \( G_{794,79400} \). This graph has 794 vertices, 79400 edges; and contains

\[
79400 - 794 = 78606 \text{ chords.}
\]

It will be interesting to find out how good is this lowest value of \( k \). We will therefore find the maximum number of (odd) chords that the 794-gon can contain. It is

\[
\binom{794}{2} = \binom{397}{2} = 78606.
\]

This means that the graph \( G_{794,79400} \) is saturated. Thus, we have indeed found the smallest order graph of the form \( G_{r,s} \).

In the above Example (ii), the smallest order graph belonging to the family of graphs of the form \( G_{r,s} \) was found. However, it is a large graph. The natural question at this stage is the following. Can we find a smaller order graph; maybe from an entirely different family with \( \frac{a}{b} = \frac{1}{100} \)? This question motivates the material in the next section.

5. The smallest connected graphs with matchable ratio

If the matchable ratio \( \frac{a}{b} \) is 0 or 1, then the smallest connected graphs are obvious. For \( \frac{a}{b} = 0 \), the smallest connected graph is \( P_3 \). For \( \frac{a}{b} = 1 \), every edge is matchable. In this case, the smallest graph is an edge. We will therefore consider only matchable ratios which are neither 0 nor 1.

We will denote vertex and edge sets of a graph \( G \), by \( V(G) \) and \( E(G) \), respectively.

Definition

Let \( A \) and \( B \) be graphs. We will say that \( A \) is smaller than \( B \) if and only if \( |V(A)| \leq |V(B)| \) and \( |E(A)| \leq |E(B)| \) and at least one of the inequalities is strict.
Lemma 4

Let \( r \) be a positive integer such that \( r = \binom{n}{2} \), for some even positive integer \( n \). Then the smallest order connected graph with \( r \) matchable edges is \( K_n \). Otherwise, the smallest order connected graph with \( r \) matchable edges is \( G_{r,0} \), if \( r \) is even and \( r \geq 4 \); and is \( H_{r-1} \), if \( r \) is odd and \( r \geq 5 \).

Proof

If a graph has \( r \) matchable edges, then it has at least \( r \) edges. Hence, any connected graph \( G \) with \( r > 0 \) matchable edges must have the following properties.

(i) \( G \) has at least \( r \) edges.

(ii) \( G \) has an even number of vertices (since it must have a perfect matching).

Case (1) \( r\text{-even} \)

For \( r = 2 \), the smallest graph is \( P_4 \) - the chain with 4 vertices. When \( r \) is even, and greater than 2, two connected graphs satisfy the minimum value of Condition (i), i.e. every edge is matchable; and Condition (ii). They are the \( r \)-cycle and the complete graph \( K_n \), where

\[
 r = \binom{n}{2}, \quad \text{for some even positive integer } n. \]

It follows that when \( r \) is even, and greater than 2, the smallest connected unsaturated graph with \( r \) matchable edges is the \( r \)-cycle.

Case (2) \( r\text{-odd} \)

When \( r = 1 \), the smallest connected graph, must have at least two vertices, by Condition (ii). Hence it must be an edge. When \( r = 3 \), two graphs satisfy the above conditions; the chain \( P_6 \) and the chain \( P_5 \), with a pendant edge attached to its centre vertex. Both graphs are trees with six vertices. For \( r \geq 5 \), we can start off the unique smallest connected matching unsaturated graph with \( r-1 \) (≥4) matchable edges—which is \( C_{r-1} \), and then add one more matchable edge, in the "cheapest" way. The desired graph \( G \) must have at least \( r+1 \) vertices. So ideally, we would like to add one matchable edge and exactly two vertices (if possible). This can be achieved in only one way; that is, by attaching a \( P_3 \) to a \( C_{r-1} \). It follows that when \( r \) is odd, and greater than 3, the smallest connected graph with \( r \) matchable edges is an \( r \)-cycle, with a \( P_3 \) attached. Hence the result follows.

In order to construct a minimum graph with a prescribed number of matchable and non-matchable edges, we would start off with the smallest graph with the desired number of matchable edges and then add non-matchable edges, so as to minimize the number of additional vertices. Clearly, if we can add the non-matchable edges to the smallest graphs, without adding any new vertices, then the resulting graphs must be the smallest possible. This
means that our initial smallest graph should be unsaturated. Thus, a complete
graph cannot be used. Therefore, the smallest graphs with \( r \) matchable edges and
\( s \) non-matchable edges and with appropriate restriction on \( s \), are the graphs \( G_{r,s} \)
and \( H_{r,s} \) defined in Section 2. Our discussion, together with Corollaries 1.1 and
2.1, lead to the following result.

**Theorem 4**

Let \( a = 2n \) (\( n > 1 \)) and \( b \) be positive integers, such that \( a < b \) and
\[
0 < b - a \leq \left( \frac{n}{2} \right).
\]
Then the smallest connected graph of size \( b \) with \( a \) matchable
edges and with matching ratio \( \frac{a}{b} \), is the graph \( G_{a,b - a} \). If \( a = 2n + 1 \) (\( n > 1 \)),
then the smallest connected graph of size \( b \) with \( a \) matchable edges and with
matching ratio \( \frac{a}{b} \) is the graph \( H_{a - 1,b - a - 1} \).

6. Discussion

At this stage, there is still one unanswered question.

**Problem** Given \( \frac{a}{b}, (0 \leq a \leq b) \), find a smallest graph \( G \), relative to either order
of size, such that \( \pi(G) = \frac{a}{b} \).

As discussed in Section 4, for some matchable ratios \( \frac{a}{b} \), it
might be impossible for any graph with \( a \) matchable edges to have size \( b \).
Therefore, a smallest graph does not exist. So the next best thing, is to look for
the smallest graph with equal matchable ratio. This case is still unsolved. Is
the graph with 794 vertices and 79400 edges, in Example (ii), the smallest
connected graph with matchable ratio \( \frac{1}{100} \)?

Lemma 3 gives the smallest multiple of the ratio, which will
yield a cycle large enough to accommodate the necessary unmatchable edges. We
are assured that the smallest graph of the \( G_{r,s} \) and \( H_{r,s} \) forms are found. But for
the cases where \( b - a > \left( \frac{n}{2} \right) \), we are not sure that these types of graphs are the
smallest connected graphs.
7. References

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